Bosons and Fermions

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Identical Particles and Exchange Operator

In quantum mechanics, particles are in principle identical (thus, not distinguishable). We introduce exchange operator as

$$\hat{P}\psi(r_1, r_2) = \psi(r_2, r_1).$$

Since particles are identical, the Hamiltonian of a system and the exchange operator must commute each other as $[H, P] = 0$. Therefore, the eigenfunctions of the Hamiltonian should be the eigenfunctions of P simultaneously. In addition, the probability density of $\hat{P}\psi(r_1, r_2)$ should satisfy

$$|\psi(r_1, r_2)|^2 = |\psi(r_2, r_1)|^2.$$

Identical Particles and Exchange Operator

Consider the following eigenvalue equation,

$$\hat{P}\psi(r_1, r_2) = \lambda\psi(r_1, r_2).$$

Note that trivially

$$\hat{P}^2\psi(r_1, r_2) = \lambda^2\psi(r_1, r_2) = \psi(r_1, r_2),$$

and $\lambda = \pm 1$.

Hence, we have two possible wave functions: symmetric and antisymmetric.

- Symmetric: $\psi(r_2, r_1) = \psi(r_1, r_2)$.
- Antisymmetric: $\psi(r_2, r_1) = -\psi(r_1, r_2)$.

Bosons and Fermions

We call the particles as boson when their wave functions are symmetric. And, we call the particles as fermion when their wave functions are antisymmetric. Also, note that the property of boson and fermion is determined by the spin of particles.

In summary,

- Boson, Symmetric: $\psi(r_2, r_1) = \psi(r_1, r_2)$ and $s = 0, 1, 2, \cdots$.
- Fermion, Antisymmetric: $\psi(r_2, r_1) = -\psi(r_1, r_2)$ and $s = 1/2, 3/2, 5/2, \cdots$.

Exchange Forces

Consider two particles of a and b. Then the expectation value of distance between the two particles is given by

$$\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle.$$

We have three possibilities:

- Distinguishable (classical): $\psi = \psi_a(x_1)\psi_b(x_2)$
- Bosons: $\psi_+ = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) + \psi_b(x_1)\psi_a(x_2)\right]$
- Fermions: $\psi_- = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) - \psi_b(x_1)\psi_a(x_2)\right]$

Exchange Forces

- Distinguishable (classical):
  $\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b$
- Bosons: $\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b - 2|\langle x \rangle_{ab}|^2$
- Fermions: $\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b + 2|\langle x \rangle_{ab}|^2$

(-) in Boson case is the origin of covalent bond.