Assignment 1

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1 Momentum in Wave Mechanics (20 pt)

Given the wave function $\Psi(x,t)$, we can calculate the expectation value of any function of x:

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} \Psi(x,t)^* f(x)\Psi(x,t)dx. \tag{1}$$

But, the above equation does not tell us how to calculate the expectation value of the momentum $\langle p\rangle$ because we do not know what to insert between $\Psi(x,t)^*$ and $\Psi(x,t)$ when calculating $\langle p\rangle$. In order to find the momentum operator in position space, we begin with classical equations,

$$p = mv = m\frac{dx}{dt}. \tag{2}$$

Now, consider the expectation value of the momentum from the time evolution of $\langle x\rangle$,

$$\begin{aligned}
\langle p\rangle &= m\frac{d\langle x\rangle}{dt} \\
&= m\frac{d}{dt}\int_{-\infty}^{\infty} \Psi^* x\Psi dx \\
&= m\int_{-\infty}^{\infty} dx\left[\frac{\partial \Psi^*}{\partial t}x\Psi + \Psi^* x\frac{\partial \Psi}{\partial t}\right].
\end{aligned} \tag{3}$$

Then, putting the Schrödinger equation into Eq. 3, show that the momentum operator is

$$p = -i\hbar\frac{\partial}{\partial x}. \tag{4}$$

2 Classical Harmonic Oscillator (20 pt)

Consider a particle with mass m on a frictionless surface attached to a spring. The equation of motion of the particle obeys the Hooke's law:

$$F = m\frac{d^2x}{dt^2} = -kx, \tag{5}$$

where k is the spring constant. Rearranging the equation, we obtain the following second order differential equation

$$\frac{d^2x}{dt^2} = -\omega^2 x, \quad \text{where } \omega = \sqrt{\frac{k}{m}}. \tag{6}$$

Recognize that Eq. 6 is mathematically the same as the time-independent Schrödinger equation in the infinite potential well. Show that the general solution of Eq. 6 is

$$x(t) = A\sin\omega t + B\cos\omega t. \tag{7}$$

In addition, describe the position and the velocity at $t=0$ in terms of A, B, and ω.

3 Quantum Harmonic Oscillator (60 pt)

The simple harmonic oscillators in quantum mechanics is extremely important theoretically and practically. In a sense, it contains the essence of the quantum mechanics and helps to understand the angular momentum in quantum mechanics. If you are not familiar with quantum harmonic oscillator yet, now you have to review this problem seriously.

The Hamiltonian of simple harmonic oscillator is given by

$$\begin{aligned} \hat{H} &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 \\ &= \frac{p^2}{2m} + \frac{1}{2m}(m\omega x)^2 \end{aligned} \tag{8}$$

3.1 Canonical Commutation relation

First show that

$$[x, p] = xp - px = i\hbar. \tag{9}$$

3.2 Ladder Operators

Defining lowering and raising operators as

$$\begin{aligned} a_- &= \frac{1}{\sqrt{2\hbar m\omega}}(ip + m\omega x), \\ a_+ &= \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x), \end{aligned}$$

show that

$$\begin{aligned} [a_-, a_+] &= 1, \\ a_- a_+ &= \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}. \end{aligned}$$

3.3 Number Operator

Defining number operator as

$$n = a_+ a_-, \tag{10}$$

show that

$$\hat{H} = \hbar\omega\left(n + \frac{1}{2}\right). \tag{11}$$

Use the commutation relation $[a_-, a_+] = 1$.

3.4 Wave Functions

If ψ satisfy the relation $\hat{H}\psi = E\psi$ meaning that ψ is a solution of the Schrödinger equation with energy E, show that $a_+\psi$ and $a_-\psi$ satisfy the Schrödinger equation with energy $(E + \hbar\omega)$ and $(E - \hbar\omega)$, respectively.

3.5 Ground State

From the relation $a_-\psi_0 = 0$, determine the ground state of the harmonic oscillator:

$$\frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)\psi_0 = 0. \tag{12}$$

3.6 Ground State

Finally, show that the wave function and the energy of the harmonic oscillator:

$$\psi_n(x) = A_n(a_+)^n\psi_0(x), \quad \text{with} \quad E_n = \hbar\omega\left(n + \frac{1}{2}\right). \tag{13}$$