Assignment 10 (for the final exam)

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You do not need to submit this assignment. But, you have to know that these problems will appear in the final exam with high probability.

1 (Important) Hermitian Operators

Show that the eigenfunctions of a hermitian operator have the following two properties for discrete spectra.

1.1 Real Eigenvalues

Show that the eigenvalues of a hermitian operator are real.

1.2 Orthogonality

Show that the eigenfunctions belonging to different eigenvalues are orthogonal.

2 (Important) Compatible Observables

When two operators A and B commute each other ($AB = BA$), show that they can share simultaneous eigenfunctions, that is

$$\begin{aligned} A\psi &= a\psi. \\ B\psi &= b\psi. \end{aligned} \tag{1}$$

3 Ehrenfest Theorem

Show that the time evolution of the expectation value of some operator $Q(x, p, t)$ is

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle [H, Q]\rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle, \tag{2}$$

where H is the Hamiltonian.

4 Ehrenfest Theorem II

Apply Equation 2 to the special cases: (a) x and (b) p.

5 Hydrogen Atom

An electron of Hydrogen atom is in a state described by the wave function ($\psi_{n,l,m}$):

$$\frac{1}{6}\left[4\psi_{1,0,0} + 3\psi_{2,1,1} - \psi_{2,1,0} + \sqrt{10}\psi_{2,1,-1}\right]. \tag{3}$$

(a) What is the expectation value of the energy? (b) What is the expectation value of L^2? and (c) What is the expectation value of L_z?

6 (Important) Three-Dimensional Harmonic Oscillator I

Consider the three-dimensional harmonic oscillator with the potential:

$$\begin{aligned} V(r) &= \frac{1}{2}m\omega^2 r^2 \\ &= \frac{1}{2}m\omega^2(x^2 + y^2 + z^2). \end{aligned} \tag{4}$$

Show that separation of variables in Cartesian coordinates gives the energy spectrum,

$$E_n = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right). \tag{5}$$

7 Three-Dimensional Harmonic Oscillator II

Consider the three-dimensional harmonic oscillator with the potential:

$$V(r) = \frac{1}{2}m\omega^2 r^2. \tag{6}$$

Find the radial and angular equations using the separation of variables in Spherical coordinates. The energy eigenvalues in Spherical coordinates are given by

$$E_{n,\ell} = \hbar\omega\left(2n + \ell + \frac{3}{2}\right). \tag{7}$$

Check the ground and first and second excited states in Spherical coordinates and in Cartesian coordinates.

8 Angular Momenta

Show the following commutation relations: (a) $[L_x, L_y] = i\hbar L_z$, (b) $[L^2, L_z] = 0$, (c) $[L_z, x] = i\hbar y$, (d) $[L_z, z] = 0$, (e) $[L_z, p_x] = i\hbar p_y$, and (f) $[L_z, p_z] = 0$.

9 (Important) Spin and Pauli Matrices

Pauli matrices for a spin 1/2 particle are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of these three matrices. Also check the matrices are hermitian.

10 Spin and Pauli Matrices II

An electron is in the spin state

$$\chi = A\begin{pmatrix} 1 - 2i \\ 2 \end{pmatrix} \tag{8}$$

in the Pauli representation. (a) Show the constant A by normalizing χ. (b) If a measurement of S_z is made, what values will be obtained, and with what probabilities? What is the expectation value of S_z? Repeat the previous calculations for (c) S_x and (d) S_y.

11 Spin and Pauli Matrices III

A measurement of S_x for a spin 1/2 system leads to the eigenvalue $\hbar/2$. Subsequently, a measurement of $S_x \cos\theta + S_y \sin\theta$ is carried out. What is the probability that the result is $+\hbar/2$?

12 Spin and Pauli Matrices IV

Consider the spin state

$$\chi = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{9}$$

What is the probability that a measurement of $(3S_x + 4S_y)/5$ yields the value $-\hbar/2$?

13 Spin and Pauli Matrices V

What is the probability of an $S_y = 1/2$ spin if an electron enters the rotated S_y Stern-Gerlach filter with S_z spin down?

14 Identical Particles

Suppose that we have two noninteracting particles in the infinite square well. The one particle states are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right), \tag{10}$$

with the energy eigenvalues

$$E_n = \frac{\pi^2 \hbar^2}{2ma^2} n^2 = Kn^2. \tag{11}$$

Find the ground state of distinguishable particles, Boson, and Fermion. And, find the first excited state of distinguishable particles and Boson.

15 (Important) Time Independent Perturbation Theory

Consider a Hamiltonian

$$\begin{aligned} H &= H^0 + H^1 \qquad (12) \\ &= \begin{pmatrix} E_0 & 0 \\ 0 & -E_0 \end{pmatrix} + \begin{pmatrix} 0 & \lambda \\ \lambda^* & 0 \end{pmatrix}. \qquad (13) \end{aligned}$$

(a) Consider H^1 as a perturbation and calculate the energy shift (the first-order correction). (b) Consider the Hamiltonian H and calculate the exact eigenvalues.

16 An-harmonic Oscillators I

Consider for the quantum oscillator problem with the Hamiltonian,

$$\begin{aligned} H^0 &= \frac{p^2}{2m} + \frac{1}{2m}(m\omega x)^2 \\ &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2m}(m\omega x)^2. \end{aligned} \tag{14}$$

Applying perturbation

$$H^1 = \lambda x^4,$$

show that the first-order correction to the ground-state energy is

$$E^1 = \frac{3}{4}\lambda \left(\frac{\hbar}{m\omega}\right)^2. \tag{15}$$

The ground state wave function is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{(1/4)} e^{-z^2/2}, \tag{16}$$

where $z = \sqrt{\frac{m\omega}{\hbar}} x$.

17 An-harmonic Oscillators II

Consider an-harmonic oscillator with the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \lambda x^4. \tag{17}$$

Using a trial wave function

$$\psi(x,\alpha) = Ae^{-\alpha x^2/2}, \tag{18}$$

show that the best approximation of the energy of the ground state is

$$E(\alpha) = \frac{3}{8}\left(\frac{6\hbar^4\lambda}{m^2}\right)^{1/3}, \tag{19}$$

and the best approximation of the ground state corresponds to

$$\alpha = \left(\frac{6m\lambda}{\hbar^2}\right)^{1/3}. \tag{20}$$