Assignment 10 (for the final exam)

Byungjoon Min, Quantum Mechanics II

You do not need to submit this assignment. But, you have to know that these problems will appear in the final exam with high probability.

1 (Important) Hermitian Operators

Show that the eigenfunctions of a hermitian operator have the following two properties for discrete spectra.

1.1 Real Eigenvalues

Show that the eigenvalues of a hermitian operator are real.

1.2 Orthogonality

Show that the eigenfunctions belonging to different eigenvalues are orthogonal.

2 (Important) Compatible Observables

When two operators A and B commute each other ($AB = BA$), show that they can share simultaneous eigenfunctions, that is

$$\begin{aligned} A\psi &= a\psi. \\ B\psi &= b\psi. \end{aligned} \tag{1}$$

3 Ehrenfest Theorem

Show that the time evolution of the expectation value of some operator $Q(x, p, t)$ is

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle [H, Q]\rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle, \tag{2}$$

where H is the Hamiltonian.

4 Ehrenfest Theorem II

Apply Equation 2 to the special cases: (a) x and (b) p.

5 Hydrogen Atom

An electron of Hydrogen atom is in a state described by the wave function ($\psi_{n,l,m}$):

$$\frac{1}{6}\left[4\psi_{1,0,0} + 3\psi_{2,1,1} - \psi_{2,1,0} + \sqrt{10}\psi_{2,1,-1}\right]. \tag{3}$$

(a) What is the expectation value of the energy? (b) What is the expectation value of L^2? and (c) What is the expectation value of L_z?

6 (Important) Three-Dimensional Harmonic Oscillator I

Consider the three-dimensional harmonic oscillator with the potential:

$$\begin{aligned} V(r) &= \frac{1}{2}m\omega^2 r^2 \\ &= \frac{1}{2}m\omega^2(x^2+y^2+z^2). \end{aligned} \tag{4}$$

Show that separation of variables in Cartesian coordinates gives the energy spectrum,

$$E_n = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right). \tag{5}$$

7 Three-Dimensional Harmonic Oscillator I

Consider the three-dimensional harmonic oscillator with the potential:

$$V(r) = \frac{1}{2}m\omega^2 r^2. \tag{6}$$

Find the radial and angular equations using the separation of variables in Spherical coordinates. The energy eigenvalues in Spherical coordinates are given by

$$E_{n,\ell} = \hbar\omega\left(2n + \ell + \frac{3}{2}\right). \tag{7}$$

Check the ground and first and second excited states in Spherical coordinates and in Cartesian coordinates.

8 Angular Momenta

Show the following commutation relations: (a) $[L_x, L_y] = i\hbar L_z$, (b) $[L^2, L_z] = 0$, (c) $[L_z, x] = i\hbar y$, (d) $[L_z, z] = 0$, (e) $[L_z, p_x] = i\hbar p_y$, and (f) $[L_z, p_z] = 0$.

9 (Important) Spin and Pauli Matrices

Pauli matrices for a spin 1/2 particle are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of these three matrices. Also check the matrices are hermitian.

10 Spin and Pauli Matrices II

An electron is in the spin state

$$\chi = A\begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \tag{8}$$

in the Pauli representation. (a) Show the constant A by normalizing χ. (b) If a measurement of S_z is made, what values will be obtained, and with what probabilities? What is the expectation value of S_z? Repeat the previous calculations for (c) S_x and (d) S_y.

11 Spin and Pauli Matrices III

A measurement of S_x for a spin 1/2 system leads to the eigenvalue $\hbar/2$. Subsequently, a measurement of $S_x \cos\theta + S_y \sin\theta$ is carried out. What is the probability that the result is $+\hbar/2$?

12 Spin and Pauli Matrices IV

Consider the spin state

$$\chi = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}. \tag{9}$$

What is the probability that a measurement of $(3S_x + 4S_y)/5$ yields the value $-\hbar/2$?

13 Spin and Pauli Matrices V

What is the probability of an $S_y = 1/2$ spin if an electron enters the rotated S_y Stern-Gerlach filter with S_z spin down?

14 Identical Particles

Suppose that we have two noninteracting particles in the infinite square well. The one particle states are

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right), \tag{10}$$

with the energy eigenvalues

$$E_n = \frac{\pi^2\hbar^2}{2ma^2}n^2 = Kn^2. \tag{11}$$

Find the ground state of distinguishable particles, Boson, and Fermion. And, find the first excited state of distinguishable particles and Boson.

15 (Important) Time Independent Perturbation Theory

Consider a Hamiltonian

$$\begin{aligned} H &= H^0 + H^1 & (12)\\ &= \begin{pmatrix}E_0 & 0\\ 0 & -E_0\end{pmatrix} + \begin{pmatrix}0 & \lambda\\ \lambda^* & 0\end{pmatrix}. & (13)\end{aligned}$$

(a) Consider H^1 as a perturbation and calculate the energy shift (the first-order correction). (b) Consider the Hamiltonian H and calculate the exact eigenvalues.

16 An-harmonic Oscillators I

Consider for the quantum oscillator problem with the Hamiltonian,

$$\begin{aligned} H^0 &= \frac{p^2}{2m} + \frac{1}{2m}(m\omega x)^2\\ &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2m}(m\omega x)^2. \end{aligned} \tag{14}$$

Applying perturbation

$$H^1 = \lambda x^4,$$

show that the first-order correction to the ground-state energy is

$$E^1 = \frac{3}{4}\lambda\left(\frac{\hbar}{m\omega}\right)^2. \tag{15}$$

The ground state wave function is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{(1/4)} e^{-z^2/2}, \tag{16}$$

where $z = \sqrt{\frac{m\omega}{\hbar}}x$.

17 An-harmonic Oscillators II

Consider an-harmonic oscillator with the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \lambda x^4. \tag{17}$$

Using a trial wave function

$$\psi(x,\alpha) = Ae^{-\alpha x^2/2}, \tag{18}$$

show that the best approximation of the energy of the ground state is

$$E(\alpha) = \frac{3}{8}\left(\frac{6\hbar^4\lambda}{m^2}\right)^{1/3}, \tag{19}$$

and the best approximation of the ground state corresponds to

$$\alpha = \left(\frac{6m\lambda}{\hbar^2}\right)^{1/3}. \tag{20}$$