Assignment 5

Byungjoon Min, Quantum Mechanics II
(due date: October 22, 2018)

1 Matrix Representation of Simple Harmonic Oscillator [30 pt]

1.1 Matrix Elements

The Hamiltonian of simple harmonic oscillator is defined as

$$\hat{H} = \hbar\omega\left(a_+a_- + \frac{1}{2}\right),$$
$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x),$$
$$a_- = \frac{1}{\sqrt{2\hbar m\omega}}(ip + m\omega x).$$

The ladder operators a_- and a_+ satisfy

$$a_+|n\rangle = \sqrt{n+1}|n+1\rangle,$$
$$a_-|n\rangle = \sqrt{s}|n-1\rangle.$$

Then, by using orthogonality $\langle m|n\rangle = \delta_{m,n}$ show that

$$\langle m|\hat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)\delta_{m,n},$$
$$\langle m|a_+|n\rangle = \sqrt{n+1}\delta_{m,n+1},$$
$$\langle m|a_-|n\rangle = \sqrt{n}\delta_{m,n-1}.$$

1.2 Matrix Representation

If we write $\langle m|X|n\rangle$ as matrix elements X_{mn}, show that

$$\hat{H} = \hbar\omega\begin{pmatrix} 1/2 & 0 & 0 & \cdots \\ 0 & 3/2 & 0 & \cdots \\ 0 & 0 & 5/2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$a_+ = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$a_- = \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ 0 & 0 & \sqrt{2} & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Also show that the matrix representations also satisfy the relation

$$\hat{H} = \hbar\omega\left(a_+a_- + \frac{1}{2}\right).$$

1.3 Matrix Representation of x and p

First, show that

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-),$$
$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-).$$

Next, obtain matrix elements

$$\langle m|x|n\rangle, \quad \langle m|p|n\rangle.$$

and construct matrix of x and p.

2 Matrix Representation of Angular Momentum [50 pt]

2.1 Ladder Operators [10 pt]

L_+ and L_- operators are defined as

$$\begin{aligned} L_+ &= L_x + iL_y, \\ L_- &= L_x - iL_y. \end{aligned} \tag{1}$$

Assume that L_+ and L_- operators satisfy:

$$L_+|l,m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}|l,m+1\rangle,$$
$$L_-|l,m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}|l,m-1\rangle.$$

Show that

$$\langle l,m'|L_+|l,m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\delta_{m',m+1},$$
$$\langle l,m'|L_-|l,m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}\delta_{m',m-1}.$$

2.2 Matrix Representation of Ladder Operators [10 pt]

When $l = 1$, show that

$$\begin{aligned} L_+ &= \hbar\begin{pmatrix} (m',m) = (-1,-1) & (-1,0) & (-1,1) \\ (0,-1) & (0,0) & (0,1) \\ (1,-1) & (1,0) & (1,1) \end{pmatrix} = \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \\ L_- &= \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{2}$$

2.3 Matrix Representation of L_x and L_y [10 pt]

Find matrix of L_x and L_y by using Eq. 1.

2.4 Matrix Representation of L_z [10 pt]

L_z satisfies $L_z|l,m\rangle = \hbar m|l,m\rangle$. Show that $\langle l,m'|L_z|l,m\rangle = \hbar m\delta_{m',m}$ and find matrix representation of L_z.

2.5 Commutation Relation [10 pt]

By using matrices of L_x, L_y, and L_z, show that

$$[L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y. \tag{3}$$