Measurements and the Uncertainty Principle

Byungjoon Min

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Time Evolution and Collapse

There are two different types of physical processes:

- time evolution following quantum dynamics (either Schrödinger's or Heisenberg's pictures)
- measurements in which wavefunction abruptly collapses.

Measurements and Collapse

Well, these subtle issues are strongly related to the interpretation of quantum mechanics.

- We will not deal with it seriously.
- For now, just take the most population attitude to quantum mechanics, “shut up and calculate”.
- Also, just let us define that a “measurement” is the kind of things that a scientist does in the laboratory.

Statistical Interpretation

$$\int_a^b |\Psi(x,t)|^2 dx : \tag{1}$$

probability of finding the particle between a and b, at time t.

Max Born

Determinate State

Consider eigenvectors $|\psi\rangle$ corresponding to the eigenvalues q.

$$\hat{Q}|\psi\rangle = q|\psi\rangle.$$

Determinate states are eigenfunctions of $\hat{Q}$. Check the variance of $\hat{Q}$ by using an operator $\hat{Q} - \langle Q\rangle$,

$$\sigma^2 = \langle\psi|(\hat{Q} - \langle Q\rangle)^2|\psi\rangle = 0 \tag{2}$$

Expectation Value

For an eigenvector equation $\hat{Q}|n\rangle = q_n|n\rangle$.

$$\begin{aligned} c_n &= \langle n|S\rangle \\ &= \int dx \langle n|x\rangle\langle x|S\rangle \\ &= \int dx f_n(x)^* \Psi(x), \end{aligned}$$

where $\Psi(x) = \langle x|S\rangle$ and $f_n(x) = \langle x|n\rangle$. The expectation value of Q is

$$\begin{aligned} \langle Q\rangle &= \langle S|\hat{Q}|S\rangle \\ &= \sum_n \sum_m \langle S|n\rangle\langle n|Q|m\rangle\langle m|S\rangle \\ &= \sum_n \sum_m c_n^* \langle n|Q|m\rangle c_m = \sum_n \sum_m q_n c_n^* \langle n|m\rangle c_m \\ &= \sum_n \sum_m q_n c_n^* c_m \delta_{n,m} = \sum_n q_n c_n^* c_n \\ &= \sum_n q_n |c_n|^2. \end{aligned}$$

Compatible Observabales

Consider the eigenfunctions $|f_i\rangle$ corresponding to the eigenvalue a_i of the operator $\hat{A}$,

$$\hat{A}|f_i\rangle = a_i|f_i\rangle. \tag{3}$$

If this eigenfunctions will be simultaneous eigenfunctions of another operator B with the eigenvalue b_i,

$$\hat{B}|f_i\rangle = b_i|f_i\rangle, \tag{4}$$

This implies that

$$\hat{A}\hat{B}|f_i\rangle = Ab_i|f_i\rangle = ab|f_i\rangle,$$
$$\hat{B}\hat{A}|f_i\rangle = Ba_i|f_i\rangle = ab|f_i\rangle,$$
$$(\hat{A}\hat{B} - \hat{B}\hat{A})|f_i\rangle = 0.$$

Compatible Observabales

More generally, $|S\rangle = \sum_i c_i f_i$ also holds the relation,

$$\begin{aligned}\sum_i c_i(\hat{A}\hat{B} - \hat{B}\hat{A})|f_i\rangle &= (\hat{A}\hat{B} - \hat{B}\hat{A})\sum_i c_i|f_i\rangle \\ &= [A, B]|S\rangle = 0\end{aligned}$$

Therefore, two observables are compatible each other, $[A, B] = 0$.

Uncertainty Principle

We introduce an operator $\sigma_A = \hat{A} - \langle A \rangle$. The expectation value of the square of the operator is

$$\begin{aligned}\langle \sigma_A^2 \rangle &= \langle \Psi | (\hat{A} - \langle A \rangle)^2 | \Psi \rangle \\ &= \langle (\hat{A} - \langle A \rangle)\Psi | (\hat{A} - \langle A \rangle)\Psi \rangle = \langle f | f \rangle,\end{aligned}$$

where $|f\rangle = (\hat{A} - \langle A \rangle)|\Psi\rangle$. Similarly, we define

$$\langle \sigma_B^2 \rangle = \langle g | g \rangle.$$

Due to the Schwarz inequality,

$$\langle \sigma_A^2 \rangle \langle \sigma_B^2 \rangle = \langle f | f \rangle \langle g | g \rangle \geq |\langle g | f \rangle|^2. \tag{5}$$

Uncertainty Principle

$$\langle \sigma_A^2 \rangle \langle \sigma_B^2 \rangle \geq |\langle \Psi | (\hat{A} - \langle A \rangle)(\hat{B} - \langle B \rangle) | \Psi \rangle|^2$$
$$= |\langle \sigma_A \sigma_B \rangle|^2.$$

We can rewrite the term $\sigma_A \sigma_B$ as

$$\sigma_A \sigma_B = \frac{1}{2}(\sigma_A \sigma_B - \sigma_A \sigma_B) + \frac{1}{2}(\sigma_A \sigma_B + \sigma_A \sigma_B)$$
$$= \frac{1}{2}[\sigma_A, \sigma_B] + \frac{1}{2}\{\sigma_A, \sigma_B\}$$
$$= \frac{1}{2}[A, B] + \frac{1}{2}\{\sigma_A, \sigma_B\},$$

where $[\sigma_A, \sigma_B] = [A, B]$. Note that $[A, B]$ is always anti-Hermitian ($X^\dagger = -X$) and $\{A, B\}$ is Hermitian ($X^\dagger = X$). Finally, we can conclude that

$$\langle \sigma_A^2 \rangle \langle \sigma_B^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2. \quad (6)$$

Uncertainty Principle

As an example,

$$\langle \sigma_x^2 \rangle \langle \sigma_p^2 \rangle \geq \frac{1}{4} |\langle [x, p] \rangle|^2$$
$$= \frac{1}{4} |i\hbar|^2 = \frac{\hbar^2}{4}.$$

Thus, we can recover the standard uncertainty relation,

$$\langle \sigma_x \rangle \langle \sigma_p \rangle \geq \frac{\hbar}{2}.$$

Incompatible observables ($[A, B] \neq 0$) do not have common eigenvectors, so that they cannot be observed exactly at the same time.

Where to go next...

Three dimensional problems