# Spin

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# Spin

Spin ($s$) of a particle is an intrinsic angular momentum, not related to position and momentum. The theory of spin is totally dependent on the commutation relations:

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y,$$

As orbital angular momentum, the eigenvectors of $S^2$ and $S_z$ satisfy:

$$S^2|s\ m\rangle = \hbar^2 s(s+1)|s\ m\rangle, \quad S_z|s\ m\rangle = \hbar m|s\ m\rangle.$$

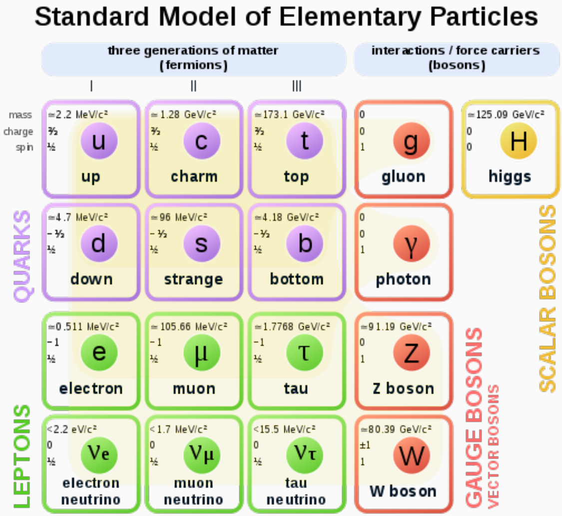
Defining the ladder operators as $S_\pm = S_x \pm iS_y$, they satisfy

$$S_\pm|s\ m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}|s\ (m\pm 1)\rangle.$$

Note that the value of $s$ and $m$ can be

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots; \quad m = -s, -s+1, \cdots, s-1, s.$$

# Spin


Standard Model of Elementary Particles
three generations of matter (fermions)
interactions / force carriers (bosons)
I
II
III
mass
charge
spin
≃2.2 MeV/c²
⅔
½
u
up
≃1.28 GeV/c²
⅔
½
c
charm
≃173.1 GeV/c²
⅔
½
t
top
0
0
1
g
gluon
≃125.09 GeV/c²
0
0
H
higgs
≃4.7 MeV/c²
−⅓
½
d
down
≃96 MeV/c²
−⅓
½
s
strange
≃4.18 GeV/c²
−⅓
½
b
bottom
0
0
1
γ
photon
≃0.511 MeV/c²
−1
½
e
electron
≃105.66 MeV/c²
−1
½
μ
muon
≃1.7768 GeV/c²
−1
½
τ
tau
≃91.19 GeV/c²
0
1
Z
Z boson
<2.2 eV/c²
0
½
νe
electron neutrino
<1.7 MeV/c²
0
½
νμ
muon neutrino
<15.5 MeV/c²
0
½
ντ
tau neutrino
≃80.39 GeV/c²
±1
1
W
W boson
QUARKS
LEPTONS
GAUGE BOSONS
VECTOR BOSONS
SCALAR BOSONS

# Spin 1/2

We define two states of spin 1/2 system as:

$$|\uparrow\rangle = \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then, $S^2$ and $S_z$ matrices are defined as

$$S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

According to the definition, the ladder operators are given by

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

# Spin 1/2

Since $S_\pm = S_x + iS_y$,

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, we define the famous Pauli spin matrices as $S = (\hbar/2)\sigma$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that the eigenvalues of the Pauli matrices are 1 and $-1$.

# Electron in a Magnetic Field

Magnetic moment of electron having a spin is

$$\mu = \gamma S. \tag{1}$$

The Hamiltonian of this particle is then

$$H = -\mu \cdot B = -\gamma B \cdot S. \tag{2}$$

Therefore, the particle with a spin tends to align parallel to $B$.

# Lamor Precession

Consider a particle with spin 1/2 in a magnetic field $B = B_0\hat{k}$ in $z$ direction. The Hamiltonian of this particle is

$$H = -\gamma B \cdot S = -\gamma B_0 S_z = -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenstates are

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \quad E_+ = -(\gamma B_0 \hbar)/2,$$

$$|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \quad E_- = +(\gamma B_0 \hbar)/2.$$

# Lamor Precession

General solution of Schrödinger equation is

$$\chi(t) = a|\uparrow\rangle e^{-iE_+t/\hbar} + b|\downarrow\rangle e^{-iE_-t/\hbar} = \begin{pmatrix} ae^{i\gamma B_0 t/2} \\ be^{-i\gamma B_0 t/2} \end{pmatrix}.$$

The initial condition of this particle can be expressed as

$$\chi(0) = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{pmatrix}.$$

Finally, we obtain

$$\chi(t) = \begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix}.$$

# Lamor Precession

The expectation value of $S_x$, $S_y$, and $S_z$ are

$$\begin{aligned}\langle S_x\rangle &= \chi(t)^\dagger S_x\chi(t) = \chi(t)^\dagger\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\chi(t)\\ &= \frac{\hbar}{2}\sin\alpha\cos(\gamma B_0 t).\\ \langle S_y\rangle &= -\frac{\hbar}{2}\sin\alpha\sin(\gamma B_0 t).\\ \langle S_z\rangle &= \frac{\hbar}{2}\cos\alpha.\end{aligned}$$

Lamor's frequency is $\omega = \gamma B_0$.

# Stern-Gerlach Experiment

An inhomogeneous magnetic field is given by

$$B = -\alpha x \hat{i} + (B_0 + \alpha z)\hat{k}.$$

Then, the Hamiltonian and force on a particle with a spin 1/2 is

$$\begin{aligned} H &= -\mu \cdot B, \\ F &= -\nabla(-mu \cdot B) = \nabla(\mu \cdot B) = \gamma\alpha(-S_x\hat{i} + S_z\hat{k}). \end{aligned}$$

Due to the Lamor precession, $\langle S_x \rangle = 0$ and the force in $z$ direction $F_z$ is given by

$$F_z = \gamma\alpha S_z.$$

# Stern-Gerlach Experiment

Suppose that the initial state of an electron is $|\uparrow\rangle$. Check $S_x$, $S_y$, and $S_z$:

$$\langle S_x \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0,$$

$$\langle S_y \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0,$$

$$\langle S_z \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1.$$

# Boson and Fermion

Spin is strongly related to quantum statistical mechanics. Particles can be classified into two kinds depending on spin values.

- Boson: $s = 0, 1, 2, \cdots$.
- Fermion: $s = 1/2, 3/2, 5/2, \cdots$.