Time Dependent Perturbation Theory

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Quantum Dynamics

Time dependent Schrödinger equation is given by

$$H|\Psi(t)\rangle = i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t}.$$

The time evolution of wave function is then

$$|\Psi(t)\rangle = \psi(r)e^{-iEt/\hbar},$$

where ψ is the solution of the time independent Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle.$$

Two Level Systems

Consider a two level system,

$$H^0|a\rangle = E_a|a\rangle,$$
$$H^0|b\rangle = E_b|b\rangle.$$

Thus in general,

$$|\Psi(t)\rangle = c_a|a\rangle e^{-iE_at/\hbar} + c_b|b\rangle e^{-iE_bt/\hbar},$$

with $|c_a|^2 + |c_b|^2 = 1$.

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Consider the Hamiltonian

$$H = H^0 + H^1(t).$$

Thus,

$$[H^0 + H^1(t)]|\Psi(t)\rangle = i\hbar \frac{\partial|\Psi(t)\rangle}{\partial t}.$$

We then have the following equation:

$$H^0 c_a|a\rangle e^{-iE_a t/\hbar} + H^0 c_b|b\rangle e^{-iE_b t/\hbar} + H^1 c_a|a\rangle e^{-iE_a t/\hbar} + H^1 c_b|b\rangle e^{-iE_b t/\hbar}$$
$$= i\hbar \left[\dot{c_a}|a\rangle e^{-iE_a t/\hbar} + \dot{c_b}|b\rangle e^{-iE_b t/\hbar} + c_a|a\rangle \frac{\partial}{\partial t} e^{-iE_a t/\hbar} + c_b|b\rangle \frac{\partial}{\partial t} e^{-iE_b t/\hbar}\right].$$

Canceling out first two terms in the left side and last two terms in the right side, we have

$$H^1 c_a|a\rangle e^{-iE_a t/\hbar} + H^1 c_b|b\rangle e^{-iE_b t/\hbar} = i\hbar \left[\dot{c_a}|a\rangle e^{-iE_a t/\hbar} + \dot{c_b}|b\rangle e^{-iE_b t/\hbar}\right].$$

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$$H^1 c_a|a\rangle e^{-iE_a t/\hbar} + H^1 c_b|b\rangle e^{-iE_b t/\hbar} = i\hbar\left[\dot{c_a}|a\rangle e^{-iE_a t/\hbar} + \dot{c_b}|b\rangle e^{-iE_b t/\hbar}\right].$$

Applying $\langle a|$, we have

$$\begin{aligned}
c_a\langle a|H^1|a\rangle e^{-iE_a t/\hbar} &+ c_b\langle a|H^1|b\rangle e^{-iE_b t/\hbar} \\
&= i\hbar\left[\dot{c_a}\langle a|a\rangle e^{-iE_a t/\hbar} + \dot{c_b}\langle a|b\rangle e^{-iE_b t/\hbar}\right] \\
&= i\hbar\dot{c_a} e^{-iE_a t/\hbar}.
\end{aligned}$$

Then,

$$\dot{c_a} = \frac{1}{i\hbar}\left[c_a\langle a|H^1|a\rangle + c_b\langle a|H^1|b\rangle e^{-i(E_b-E_a)t/\hbar}\right].$$

Similarly applying $\langle b|$, we obtain

$$\dot{c_b} = \frac{1}{i\hbar}\left[c_b\langle b|H^1|b\rangle + c_a\langle b|H^1|a\rangle e^{i(E_b-E_a)t/\hbar}\right].$$

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Typically, diagonal terms in the perturbation are zero, so we have

$$\dot{c_a} = \frac{1}{i\hbar}\langle a|H^1|b\rangle e^{-i\omega_o t/\hbar} c_b,$$
$$\dot{c_b} = \frac{1}{i\hbar}\langle b|H^1|a\rangle e^{i\omega_o t/\hbar} c_a.$$

where $\omega_o = (E_b - E_a)/\hbar$.

Assuming the initial condition $c_a(0) = 1$ and $c_b(0) = 0$, the first order approximation is

$$\dot{c_a} = 0,$$
$$\dot{c_b} = \frac{1}{i\hbar}\langle b|H^1|a\rangle e^{i\omega_o t/\hbar}.$$

Then, c_b can be obtained as

$$c_b = \frac{1}{i\hbar}\int_0^t \langle b|H^1(t')|a\rangle e^{i\omega_o t'/\hbar} dt'.$$

Interaction to Electromagnetic Field

Considering an external electric field $E = E_0 \cos(\omega t)\hat{k}$, we have the perturbation as

$$H^1 = -qEz\cos\omega t.$$

When $\omega_0 \approx \omega$, the transition probability from the state a to b is

$$\begin{aligned} P_{a\to b}(t) &= |c_b(t)|^2 \\ &\approx \left(\frac{|p|E_0}{\hbar}\right)^2 \frac{\sin^2[(\omega_o - \omega)t/2]}{(\omega_o - \omega)^2}. \end{aligned}$$

where $p = q\langle b|z|a\rangle$.