Vector Spaces

Byungjoon Min

August 26, 2018

Vector Spaces

A vector space consists of a set of vectors and a set of scalars, which is closed under two operations: vector addition and scalar multiplication.

- $|A\rangle + |B\rangle = |C\rangle$.
- $|A\rangle + |B\rangle = |B\rangle + |A\rangle$. (commutative)
- $(|A\rangle + |B\rangle) + |C\rangle = |A\rangle + (|B\rangle + |C\rangle)$. (associative)
- $|A\rangle + |0\rangle = |A\rangle$. (null vector)
- $|A\rangle + |-A\rangle = |0\rangle$. (inverse vector)
- $z|A\rangle = |B\rangle$.
- $z(|A\rangle + |B\rangle) = z|A\rangle + z|B\rangle$. (distributive)
- $(z + w)|A\rangle = z|A\rangle + w|A\rangle$. (distributive)

Examples of Vector Space

- Functions: f(x)
- Vectors: $|A\rangle \doteq \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix}$

Bra, Ket, and Dual Spaces

Suppose vectors that satisfy the axioms of vector spaces,

- Ket $|A\rangle$
- Bra $\langle B|$

We consider dual spaces between Bra and Ket,

- $|A\rangle \longleftrightarrow \langle A|$
- $|A\rangle + |B\rangle \longleftrightarrow \langle A| + \langle B|$
- $z|A\rangle \longleftrightarrow \langle A|z^*$

Component Representations

$$|A\rangle \doteq \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \quad \langle A| \doteq \left(\alpha_1^*,\ \alpha_2^*,\ \alpha_3^*\right)$$

$$|A\rangle + |B\rangle \doteq \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3 \end{pmatrix}, \quad \langle A| + \langle B| \doteq \left(\alpha_1^* + \beta_1^*,\ \alpha_2^* + \beta_2^*,\ \alpha_3^* + \beta_3^*\right)$$

$$z|A\rangle \doteq \begin{pmatrix} z\alpha_1 \\ z\alpha_2 \\ z\alpha_3 \end{pmatrix}, \quad \langle A| \doteq \left(z^*\alpha_1^*,\ z^*\alpha_2^*,\ z^*\alpha_3^*\right)$$

Inner Product

Consider the inner (dot) product of two vectors in three dimensions,

$$\begin{aligned} \vec{v}_1 &= (x_1, y_1, z_1), \\ \vec{v}_2 &= (x_2, y_2, z_2), \\ \vec{v}_1 \cdot \vec{v}_2 &= x_1 x_2 + y_1 y_2 + z_1 z_2. \end{aligned} \tag{1}$$

Let us think a more general case,

$$|A\rangle \doteq \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}, \quad |B\rangle \doteq \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

Inner Product: Bracket (Bra + Ket)

$$|A\rangle \doteq \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}, \quad |B\rangle \doteq \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\langle B|A\rangle \doteq \beta_1^*\alpha_1 + \beta_2^*\alpha_2 + \beta_3^*\alpha_3 + \cdots + \beta_n^*\alpha_n. \tag{2}$$

$$\langle B|A\rangle = \langle A|B\rangle^*$$

$$\langle C|\left(|B\rangle + |A\rangle\right) = \langle C|A\rangle + \langle C|B\rangle \quad \text{linearity}$$

$$\langle A|A\rangle = 1: \quad \text{normalized vector}$$

$$\langle B|A\rangle = 0: \quad \text{orthogonal vector}$$

Decomposition

We introduce a set of orthonormal bases $|i\rangle$

$$|A\rangle = \sum_i \alpha_i |i\rangle. \tag{3}$$

Then, we can derive some useful relations

$$\begin{aligned}\langle j|A\rangle &= \langle j|\sum_i \alpha_i |i\rangle = \sum_i \alpha_i \langle j|i\rangle \\ &= \sum_i \alpha_i \delta_{ij} = \alpha_j.\end{aligned}$$

$$|j\rangle\langle j|A\rangle = \alpha_j |j\rangle : \quad \text{projection operator}$$

$$|A\rangle = \sum_i \langle i|A\rangle|i\rangle = \sum_i |i\rangle\langle i|A\rangle.$$

$$\mathbb{1} = \sum_i |i\rangle\langle i| : \quad \text{identity operator}$$

Hilbert Space

- Mathematically, a Hilbert space is a complete inner product space, a vector space which has the structure of an inner product and completeness. **So what**?
- For physicists, a Hilbert space is a set of wave functions. Thus, wave functions live in Hilbert space.

$$\langle B|A\rangle = \sum_{i=1}^{n} \beta_i^* \alpha_i.$$

$$\sum \leftrightarrows \int dx$$

$$\langle f|g\rangle = \int_a^b f(x)^* g(x) dx,$$

where

$$\int_a^b |f(x)|^2 dx < \infty.$$

Where to go next...

Observavabales, Operators, and Matrices.