Bose-Einstein and Fermi-Dirac Distribution

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Identical Particles and Exchange Operator

In quantum mechanics, particles are in principle identical (thus, not distinguishable). We introduce exchange operator as

$$\hat{P}\psi(r_1, r_2) = \psi(r_2, r_1).$$

Since particles are identical, the Hamiltonian of a system and the exchange operator must commute each other as $[H, P] = 0$. Therefore, the eigenfunctions of the Hamiltonian should be the eigenfunctions of P simultaneously. In addition, the probability density of $\hat{P}\psi(r_1, r_2)$ should satisfy

$$|\psi(r_1, r_2)|^2 = |\psi(r_2, r_1)|^2.$$

Identical Particles and Exchange Operator

Consider the following eigenvalue equation,

$$\hat{P}\psi(r_1, r_2) = \lambda\psi(r_1, r_2).$$

Note that trivially

$$\hat{P}^2\psi(r_1, r_2) = \lambda^2\psi(r_1, r_2) = \psi(r_1, r_2),$$

and $\lambda = \pm 1$.
Hence, we have two possible wave functions: symmetric and antisymmetric.

- Symmetric: $\psi(r_2, r_1) = \psi(r_1, r_2)$.
- Antisymmetric: $\psi(r_2, r_1) = -\psi(r_1, r_2)$.

Boson and Fermion

We call the particles as boson when their wave functions are symmetric. And, we call the particles as fermion when their wave functions are antisymmetric. Also, note that the property of boson and fermion is determined by the spin of particles.
In summary,

- Boson, Symmetric: $\psi(r_2, r_1) = \psi(r_1, r_2)$ and $s = 0, 1, 2, \cdots$.
- Fermion, Antisymmetric: $\psi(r_2, r_1) = -\psi(r_1, r_2)$ and $s = 1/2, 3/2, 5/2, \cdots$.

Grand Canonical Ensemble

We slightly modify grand partition function as

$$Q(T, V, \mu) = \sum_k e^{-\beta(\epsilon_k - \mu) n_k}.$$

We consider each eigenstate as being populated independently from other eigenstates and exchanging particles with the external bath. We also define grand free energy as

$$\Phi = -kT \log Q.$$

Then, the number of particles in state k is

$$\langle n \rangle = -\frac{\partial \Phi}{\partial \mu}.$$

Bose-Einstein Distribution

For bosons, grand partition function is

$$\begin{aligned} Q_{BE} &= \sum_{n_k=0}^{\infty} e^{-\beta(\epsilon_k-\mu)n_k} \\ &= \sum_{n_k=0}^{\infty} \left(e^{-\beta(\epsilon_k-\mu)}\right)^{n_k} \\ &= \frac{1}{1-e^{-\beta(\epsilon_k-\mu)}}. \end{aligned}$$

The boson grand free energy is then

$$\Phi_k = -kT\log Q_{BE} = kT\log\left(1-e^{-\beta(\epsilon_k-\mu)}\right).$$

The number of bosons in state k is

$$\langle n_k\rangle_{BE} = -\frac{\partial\Phi_k}{\partial\mu} = \frac{1}{e^{\beta(\epsilon_k-\mu)}-1}.$$

Fermi-Dirac Distribution

For fermions, grand partition function is

$$Q_{FD} = \sum_{n_k=0}^{1} e^{-\beta(\epsilon_k - \mu)n_k} \\ = 1 + e^{-\beta(\epsilon_k - \mu)}.$$

The fermion grand free energy is then

$$\Phi_k = -kT \log Q_{FD} = -kT \log\left(1 + e^{-\beta(\epsilon_k - \mu)}\right).$$

The number of bosons in state k is

$$\langle n_k \rangle_{FD} = -\frac{\partial \Phi_k}{\partial \mu} = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}.$$

Maxwell-Boltzmann Distribution

For classical but indistinguishable particles, we can derive similarly

$$Q_{MB} = \sum_N \frac{1}{N!} \left(\sum_k e^{-\beta\epsilon_k} \right)^N e^{N\beta\mu}$$
$$= \Pi_k \exp\left(e^{-\beta(\epsilon_k - \mu)} \right).$$

The grand free energy for a single particle is then

$$\Phi_k = -kTe^{-\beta(\epsilon_k - \mu)}.$$

The number of particles in state k is

$$\langle n_k \rangle_{MB} = -\frac{\partial \Phi_k}{\partial \mu} = e^{-\beta(\epsilon_k - \mu)}.$$

BE, FD, and MB Distribution

$$\langle n \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} + \alpha},$$

- $\alpha = -1$: Bose-Einstein Distribution
- $\alpha = +1$: Fermi-Dirac Distribution
- $\alpha = 0$: Maxwell-Boltzmann Distribution