Density of States

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BE, FD, and MB Distribution

$$\langle n_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} + \alpha},$$

- $\alpha = -1$: Bose-Einstein Distribution
- $\alpha = +1$: Fermi-Dirac Distribution
- $\alpha = 0$: Maxwell-Boltzmann Distribution

For quantum gases,

$$\langle n_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} \pm 1},$$

where $-$ for bosons and $+$ for fermions.

Number of Particles and Energy

Then, the number of total particles and energy are then

$$N = \sum_k \langle n_k \rangle = \int_0^\infty \frac{g(E)dE}{e^{\beta(E-\mu)} \pm 1},$$

$$E = \sum_k \langle n_k \rangle E_k = \int_0^\infty \frac{Eg(E)dE}{e^{\beta(E-\mu)} \pm 1},$$

where $g(E)$ is the density of states. And, defining the fugacity $z = e^{\beta\mu}$,

$$N = \sum_k \langle n_k \rangle = \int_0^\infty \frac{g(E)dE}{z^{-1}e^{\beta E} \pm 1},$$

$$E = \sum_k \langle n_k \rangle E_k = \int_0^\infty \frac{Eg(E)dE}{z^{-1}e^{\beta E} \pm 1}.$$

Density of States (Free Particles)

Consider particles in a box with $V = L \times L \times L$.

$$\Psi(x,y,z) = \left(\frac{2}{L}\right)^{3/2} \sin(k_x x)\sin(k_y y)\sin(k_z z),$$

$$E_n = \frac{\hbar^2 k^2}{2m}, \quad k_x = \frac{n_x \pi}{L}, \quad k_y = \frac{n_y \pi}{L}, \quad k_z = \frac{n_z \pi}{L},$$

where n_x, n_y, and n_z are integer. Thus,

$$k^2 = \frac{\pi^2}{L^2}(n_x^2 + n_y^2 + n_z^2) = \frac{\pi^2}{L^2} r^2.$$

where $r^2 = n_x^2 + n_y^2 + n_z^2 = \left(\frac{kL}{\pi}\right)^2$.

Density of States (Free Particles)

The total number of states $G(k)$ with k-vectors of magnitude between 0 and k is one-eighth (1/8) of the volume of the sphere of radius r,

$$G(k) = \frac{1}{8} \times \frac{4}{3}\pi r^3 = \frac{1}{8} \times \frac{4}{3}\pi \left(\frac{kL}{\pi}\right)^3.$$

And, the density of states is

$$g(k)dk = \frac{dG(k)}{dk}dk = \frac{1}{8} \times 4\pi k^2 \left(\frac{L}{\pi}\right)^3 dk = \frac{Vk^2}{2\pi^2}dk.$$

Density of States (Electron)

Electrons in solids (metals) are fermions and are not fixed in number. The density of states in k-space is the same as

$$g(k)dk = \frac{Vk^2}{2\pi^2}dk.$$

For the density of state in energy space,

$$\epsilon = \frac{\hbar^2 k^2}{2m}, \quad d\epsilon = \frac{\hbar^2}{2m}(2k)dk,$$

$$g(\epsilon)d\epsilon = 2 \times \frac{Vk^2}{2\pi^2}\frac{m}{\hbar^2 k}d\epsilon = 2 \times \frac{V}{2\pi^2}\frac{m}{\hbar^2}kd\epsilon$$

$$= 2 \times \frac{V}{2\pi^2}\frac{m}{\hbar^2}\frac{\sqrt{2m\epsilon}}{\hbar}d\epsilon = 2 \times \frac{mV}{2\pi^2\hbar^3}\sqrt{2m\epsilon}d\epsilon,$$

where 2 is for two different spins.

Density of States (Photon)

The density of states in k-space is (we can derive it from Maxwell equations)

$$g(k)dk = \frac{Vk^2}{2\pi^2}dk.$$

In order to obtain the density of state for photons in energy space,

$$\epsilon = \hbar\omega = \hbar ck, \quad d\epsilon = c\hbar dk,$$

$$g(\epsilon)d\epsilon = 2 \times \frac{V[\epsilon/(c\hbar)]^2}{2\pi^2}\frac{1}{c\hbar}d\epsilon = 2 \times \frac{V\epsilon^2}{2\pi^2(c\hbar)^3}d\epsilon,$$

where 2 is for two different polarizations. And, for the density of state in frequency space,

$$\omega = ck, \quad d\omega = cdk,$$

$$g(\omega)d\omega = 2 \times \frac{V\omega^2}{2\pi^2c^3}d\omega.$$

Density of States (Phonon)

Phonons are quantized thermal waves in a solid. They are bosons and are not fixed in number. The density of states in k-space is

$$g(k)dk = \frac{Vk^2}{2\pi^2}dk.$$

For the density of state in frequency space,

$$\epsilon = \hbar\omega, \quad \omega = vk, \quad d\omega = vdk,$$

$$g(\omega)d\omega = 3 \times \frac{V\omega^2}{2\pi^2 v^3}d\omega,$$

where 3 is for two different polarizations (2 transverse and 1 longitudinal).