Assignment 4

Byungjoon Min, Statistical Mechanics
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1 Ideal Gases [30 pt]

1.1 Sackur-Tetrod Formula [10 pt]

For the ideal gas, the number of states with two corrected factors $N!$ and h is given by

$$\Omega = V^N \frac{(2\pi m E)^{3N/2}}{\Gamma(3N/2+1)N!h^{3N}}. \tag{1}$$

The entropy of the ideal gas is given by $S = k_B \log \Omega$. Combining these two facts, derive the Sackur-Tetrode formula,

$$S(N,V,E) = Nk_B \log\left[\frac{V}{N}\left(\frac{4\pi m E}{3Nh^2}\right)^{3/2}\right] + \frac{5}{2}Nk_B. \tag{2}$$

1.2 Equipartition Theorem [10 pt]

Show that

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N} = \frac{3Nk_B}{2E}. \tag{3}$$

1.3 Equation of state [10 pt]

Show that

$$\frac{P}{T} = \left(\frac{\partial S}{\partial V}\right)_{E,N} = \frac{Nk_B}{V}. \tag{4}$$

2 Harmonic Oscillators [30 pt]

Consider N distinguishable classical harmonic oscillators with mass m and frequency w in one dimension. The Hamiltonian of the oscillators is given by

$$H = \sum_i^N \left[\frac{p_i^2}{2m} + \frac{1}{2}mw^2 q_i^2\right]. \tag{5}$$

2.1 Number of States [10 pt]

The number of states for $H \leq E$ is given by

$$\begin{aligned}\Sigma(E) &= \frac{1}{h^N}\int_{H\leq E} d^N q d^N p \\ &= \frac{1}{h^N}\left(\frac{1}{mw}\right)^N \int_{\sum_i x_i^2 + p_i^2 \leq 2mE} d^N x d^N p\end{aligned}$$

where $x = mwq$. By using the fact that the volume of n-dimensional sphere with radius R is $V_n(R) = R^N \frac{\pi^{n/2}}{(n/2)!}$ Show that

$$\begin{aligned}\Sigma(E) &= \frac{1}{h^N}\left(\frac{1}{mw}\right)^N \frac{\pi^N}{N!}(2mE)^N \\ &= \frac{1}{N!}\left(\frac{E}{\hbar w}\right)^N . \end{aligned} \tag{6}$$

Then, obtain the number of states as

$$\begin{aligned}\Omega(E) &\approx \frac{d\Sigma(E)}{dE} \times E \\ &= \frac{1}{(N-1)!}\left(\frac{E}{\hbar w}\right)^N . \end{aligned} \tag{7}$$

2.2 Entropy [10 pt]

Show that the Entropy of harmonic oscillators

$$\begin{aligned} S &= k_B \log \Omega \\ &= N k_B \left[1 + \log\left(\frac{E}{N\hbar w}\right)\right] . \end{aligned} \tag{8}$$

2.3 Temperature, Pressure, Specific Heat [10 pt]

Show that

$$\begin{aligned}\frac{1}{T} &= \left(\frac{\partial S}{\partial E}\right)_{V,N} = \frac{N k_B}{E}, \\ \frac{P}{T} &= \left(\frac{\partial S}{\partial V}\right)_{E,N} = 0, \\ C &= \frac{\partial E}{\partial T} = N k_B . \end{aligned}$$