Ising Model

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Ising Model and Magnetism

Ising model has a lattice of N sites with a single, two-state degree of freedom s_i spin on each site that take values ± 1. Taking into account pairwise interactions between nearest neighbors, the Hamiltonian of the Ising model is given by

$$H = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i. \tag{1}$$

- $\langle ij \rangle$ represents the sum over all pairs of nearest neighbor sites
- J is the coupling strength between the neighbor sites
- When $J > 0$ (the usual case), the model favors parallel spins (ferromagnetic interaction).
- h stands for an external field.

Ising Model and Magnetism

$$H = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i.$$

- At low T, the spins will organize to form a ferromagnetic phase (mostly pointing up or mostly pointing down).
- At high T, the spins will fluctuate wildly in a paramagnetic phase due to the thermal fluctuation (entropy).

Therefore, we find a phase transition of magnetization per spin $m(T) = \frac{1}{N} \sum_i s_i$ between the ferromagnetic and paramagnetic phases.

Solving the Ising Model

How to solve the Ising model?

- 0 dimension: It is a just two-level system. Solve it using the partition function.
- 1 dimension: Solve the Ising model analytically as Ising did.
- 2 dimension: Be smart and try Onsager's solution.
- 3 dimension: Be the first one who solves this problem.
- Any dimension: Perform the Monte Carlo simulation.
- Attempt some approximations such as the mean-field theory.

Mean-Field Theory

Ignoring all the fluctuation of spins, we can regard the interaction terms as an effective field. Thus,

$$\begin{aligned} s_i s_j &= [(s_i - m) + m][(s_j - m) + m] \\ &= (s_i - m)(s_j - m) + (s_i - m)m + (s_j - m)m + m^2 \\ &\approx (s_i + s_j)m - m^2, \end{aligned}$$

assuming $\delta s_i = (s_i - m)$ is small. Then,

$$\begin{aligned} H &\approx -J \sum_{\langle ij \rangle} [m(s_i + s_j) - m^2] - h \sum_i s_i \\ &= -J \sum_i \left[\frac{1}{2} zm(s_i + s_i) - \frac{1}{2} zm^2 \right] - h \sum_i s_i \\ &= \frac{1}{2} JNzm^2 - (Jzm + h) \sum_i s_i \end{aligned}$$

where z is the number of neighbor sites.

Mean-Field Theory

Applying the mean-field theory,

$$H \approx \frac{1}{2} JNzm^2 - (Jzm + h) \sum_i s_i$$

and then the partition function is

$$\begin{aligned} Z &= \sum_{\{s_i\}} e^{-\beta H} \\ &= e^{-\beta Jzm^2 N/2} \{2 \cosh[\beta(Jzm + h)]\}^N \end{aligned}$$

The magnetization is

$$\begin{aligned} m = \frac{1}{N} \sum_i s_i &= \frac{1}{\beta N} \frac{\partial \log Z}{\partial h} \\ &= \tanh[\beta(Jzm + h)]. \end{aligned}$$

Mean-Field Theory

By solving the self-consistency equation of the magnetization graphically, we can obtain m,

$$m = \tanh[\beta(Jzm + h)].$$

At $h = 0$,

- $\beta Jz < 1$: paramagnetic phase ($m = 0$).
- $\beta Jz > 1$: ferromagnetic phase ($m = \pm m_0$).

Finally, we find a typical second-order phase transition between paramagnetic and ferromagnetic phases.

Critical Phenomena

Near $x \approx 0$, we can expand

$$\tanh(x) = x - \frac{1}{3}x^3 + \cdots .$$

At $h = 0$ and T_c,

$$\begin{aligned} m &= \tanh(\beta J z m) \\ &\approx \beta_c J z m \end{aligned}$$

Finally, we have the critical temperature

$$T_c = \frac{Jz}{k}$$

Critical Phenomena

Near T_c,

$$m \approx \beta_c J z m - \frac{1}{3}(\beta J z m)^3$$

and

$$m^2 = \frac{3}{(\beta J z)^3}(\beta J z - 1) = 3\frac{T^2}{T_c^2}\left(\frac{T_c}{T} - 1\right)$$
$$= 3\left(\frac{T}{T_c}\right)^2\left(\frac{T_c - T}{T_c}\right).$$

Finally, we obtain the relation

$$m \sim \left(\frac{T_c - T}{T_c}\right)^{1/2}.$$

(Advanced) Critical Exponents

Defining $\tau = \frac{T-T_c}{T_c}$,

- $C = \frac{\partial E}{\partial T} \sim |\tau|^{-\alpha}$.
- $m \sim (-\tau)^{\beta}$.
- $\chi = \frac{\partial m}{\partial h} \sim |\tau|^{-\gamma}$.
- $h \sim |\tau|^{\delta}$.
- $\xi \sim |\tau|^{-\nu}$.
- $G(r) \sim \frac{1}{r^{d-2+\eta}}$.

(Advanced) Universality Classes

For the models with short-range interactions, the critical exponents depend only on the dimensionality of space and the symmetry of the order parameter. For instance, $(\alpha, \beta, \gamma, \delta, \nu, \eta)$ are

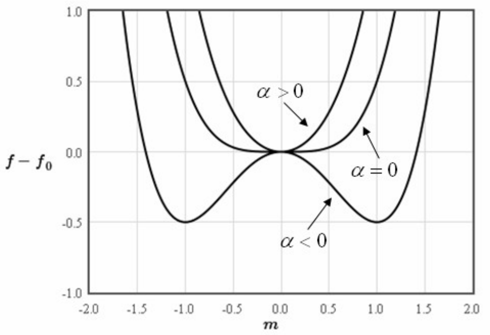
- 2-d Ising: (0, 1/8, 7/4, 15, 1, 1/4).
- 3-d Ising: (0.10, 0.33, 1.24, 4.8, 0.63, 0.04).
- mean-field: (0, 1/2, 1, 3, 1/2, 0).

(Advanced) Landau Theory

Assuming the landau free energy in terms of the magnetization as

$$f = f_0 + \alpha m^2 + a_4 m^4,$$

with $a_4 > 0$.



(Advanced) Landau Theory

Rearranging the equation,

$$f = f_0 + \alpha\tau m^2 + a_4 m^4.$$

The steady states are given by

$$\frac{\partial F}{\partial m} = 2\alpha\tau m + 4a_4 m^3 = 0$$

Rearranging the equation, we have

$$m \sim (-\tau)^{1/2}.$$