Photon and Black-Body Radiation

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Bose-Einstein Distribution and Density of States

Now, we have all ingredients to analyze photon statistics,

$$\langle n \rangle = \frac{1}{e^{\beta(\epsilon-\mu)} - 1},$$

$$\epsilon = \int_0^\infty \frac{\epsilon g(\epsilon) d\epsilon}{e^{\beta(\epsilon-\mu)} \pm 1},$$

$$g(\omega) d\omega = 2 \times \frac{V\omega^2}{2\pi^2 c^3} d\omega.$$

Photon Gases

Photons can be created or destroyed freely, meaning that the chemical potential μ of photons is zero. Thus, the number of photons per eigenstate with frequency ω is

$$\begin{aligned}\langle n_\omega \rangle &= \frac{1}{e^{\beta(\epsilon_w - \mu)} - 1} \\ &= \frac{1}{e^{\beta\hbar\omega} - 1}.\end{aligned}$$

Note that the same form is derived from the statistics of quantum harmonic oscillators. The number of photons in a given frequency interval is given by

$$\begin{aligned}n_\omega d\omega &= g(\omega)n(\omega)d\omega \\ &= 2 \times \frac{V\omega^2}{2\pi^2 c^3} \times \frac{1}{e^{\beta\hbar\omega} - 1} d\omega.\end{aligned}$$

Black-Body Radiation

The energy per unit frequency interval is

$$U(\omega)d\omega = \hbar\omega \times 2 \times \frac{V\omega^2}{2\pi^2 c^3} \times \frac{1}{e^{\beta\hbar\omega} - 1} d\omega.$$

The total energy of the radiation in a box is then

$$U = \frac{V\hbar}{\pi^2 c^3} \int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega} - 1} d\omega.$$

This is Planck's formula for black-body radiation.

Black-Body Radiation

Let $x = \beta\hbar\omega$, we obtain

$$U = \frac{V\hbar}{\pi^2 c^3}\left(\frac{1}{\beta\hbar}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1} dx.$$

This integral is not simple, but fortunately we can obtain the exact value as $\int_0^\infty \frac{x^3}{e^x-1} dx = \frac{\pi^4}{15}$. Therefore, the energy is

$$U = \frac{\pi^2 V k_B^4}{15\hbar^3 c^3} T^4.$$

Thus, the energy flux follows Stefan-Boltzmann's Law:

$$\eta = \sigma T^4. \tag{1}$$

Black-Body Radiation

Planck's formula for black-body radiation is

$$\frac{U}{V} = \frac{\hbar}{\pi^2 c^3} \int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega} - 1} d\omega = \int_0^\infty u_\omega d\omega.$$

We define u_ω as the spectral energy density,

$$u_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}.$$

- At low frequency: $e^{\beta\hbar\omega} \approx 1 + \beta\hbar\omega$, we recover the Rayleigh-Jeans formula (and also equipartition theorem)

$$u_\omega \approx \frac{1}{\pi^2 c^3} \frac{\omega^2}{\beta} \sim k_B T.$$

- At high frequency: $1/(e^{\beta\hbar\omega} - 1) \approx e^{-\beta\hbar\omega}$, we can remedy the ultraviolet catastrophe

$$u_\omega \approx \frac{\hbar}{\pi^2 c^3} \omega^3 e^{-\beta\hbar\omega}.$$