Probability and Statistics

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Outline

What is Probability?

The asymptotic frequency of success in the limit of an infinite number of trials

$$p = \lim_{N\to\infty} \frac{N_s}{N}. \tag{1}$$

The probabilities must satisfy the conditions that

$$0 \leq P(i) \leq 1, \tag{2}$$

for all i. An impossible event has probability zero and a certain event has probability 1. The normalization condition of the probability distribution is

$$\sum_i P(i) = 1. \tag{3}$$

Multivariate Probability

The conditional probability is the probability of an event a, given that an event b has occurred,

$$P(a|b) = \frac{P(a,b)}{P(b)}. \tag{4}$$

When two variables are independent one another,

$$\begin{aligned} P(a,b) &= P(a)P(b), \\ P(a|b) &= \frac{P(a,b)}{P(b)} = \frac{P(a)P(b)}{P(b)} = P(a). \end{aligned}$$

Mean, Variance, and Standard Deviation

The n-th moment of x can be defined as

$$\langle x^n \rangle = \sum_i x^n P(i). \tag{5}$$

Mean, variance, and standard deviation can be expressed as the moments of the probability distribution

$$\langle x \rangle = \sum_i x P(i),$$
$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2,$$
$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}.$$

Binomial Distribution

The binomial distribution is given by

$$P(n|N) = \binom{N}{n} p^n (1-p)^{N-n} \tag{6}$$
$$= \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}.$$

It represents the probability distribution of the number of successes (yes-no question) with the probability p, out of total N trials.

Coin Tossing

- Prob(Head)=1/2 and Prob(Tail)=1/2
- Let us flip a coin N times.
- What is the probability of getting n heads?

$$P(n|N) = \binom{N}{n} p^n (1-p)^{N-n} \tag{7}$$
$$= \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}.$$

Stirling's Approximation for $N!$

For large N,

$$N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N, \tag{8}$$
$$\log N! \approx N \log N - N.$$

$$\log N! \approx \int_1^N \log x dx = [x \log x - x]_1^N$$
$$= N \log N - N + 1.$$

Continuous probability theory

Given the probability density $P(x)$, the probability of finding x in an interval $[a, b]$ is

$$P([a, b]) = \int_a^b P(x)dx. \tag{9}$$

The n-th moment of the distribution is

$$\langle x^n \rangle = \int_{-\infty}^{\infty} x^n P(x)dx. \tag{10}$$

Dirac Delta Functions

$$\delta(x - x_0) = \begin{cases} +\infty, & x = x_0 \\ 0, & x \neq x_0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1.$$

$$\int_{-\infty}^{\infty} f(x)\delta(x - \alpha) dx = f(\alpha).$$

$$\delta(x - \alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip(x-\alpha)} dp.$$

Gaussian (Normal) Distribution

The Gaussian distribution with the mean x_0 and the standard deviation σ is given by

$$g(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right]. \tag{11}$$

- Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}. \tag{12}$$

Gaussian (Normal) Distribution

- Gaussian approximation to the Binomial Distribution
- Central limit theorem:

  Sum of independent and identically distributed random variables with finite variance σ^2 is converging to the normal distribution with variance σ^2 in the limit $N \to \infty$.

  We will prove it later.

Where to go next. . .

Let us go to random walk.